

Refined Solutions of Time Inhomogeneous Optimal Stopping Games via Dirichlet Form

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Abstract

The properties of value functions of time inhomogeneous optimal stopping problem and zero-sum game (Dynkin game) are studied through time dependent Dirichlet form. Under the absolute continuity condition on the transition function of the underlying diffusion process and some other assumptions, the refined solutions without exceptional starting points are proved to exist, and the value functions, which are finely and cofinely continuous, are characterized as the solutions of some variational inequalities.

Key words: Time inhomogeneous Dirichlet form, Optimal stopping, Dynkin game, Variational inequality

AMS subject classifications: 31C25, 49J40, 60G40, 60J60

1 Introduction

Let $\mathbf{M} = (X_t, P_{(s,x)})$ be a diffusion process on a locally compact separable metric space \mathbb{X} . For two finely continuous functions g, h on $[0, \infty) \times \mathbb{X}$ and a constant $\alpha > 0$, define the following return functions of optimal stopping games:

$$J_{(s,x)}(\sigma) = E_{(s,x)}(e^{-\alpha\sigma}g(s+\sigma, X_{s+\sigma})) \quad (1)$$

$$J_{(s,x)}(\tau, \sigma) = E_{(s,x)}[e^{-\alpha(\tau \wedge \sigma)}(g(s+\sigma, X_{s+\sigma})I_{\tau > \sigma} + h(s+\tau, X_{s+\tau})I_{\tau \leq \sigma})] \quad (2)$$

The values of the stopping games are defined as $\tilde{e}_g = \sup_{\sigma} J_{(s,x)}(\sigma)$ and $\tilde{w} = \sup_{\sigma} \inf_{\tau} J_{(s,x)}(\tau, \sigma)$, respectively. This kind of optimal stopping problems have been continually developed due to its broad application in finance, resource control or production management.

In the time homogeneous case, where \mathbf{M}, g, h are all time homogenous, it is well known that \tilde{e}_g is a quasi continuous version of the solution of a variational inequality problem formulated in terms of the Dirichlet form, see Nagai [6]. This result was successfully extended

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by Zabczyk [12] to Dynkin game (zero sum game) where \tilde{w} was shown to be the quasi continuous version of the solution of a certain variational inequality problem. In their work, there always exist an exceptional set N of starting points of \mathbf{M} . In 2006, Fukushima and Menda [3] showed that if the transition function of \mathbf{M} is absolutely continuous with respect to the underlying measure \mathbf{m} , then there does not exist the exceptional set N , and \tilde{e}_g and \tilde{w} are finely continuous with any starting point of \mathbf{M} .

However, more work is needed to extend these results to the time inhomogeneous case. Using the time dependent Dirichlet form (generalized Dirichlet form), Oshima [8] showed that under some conditions, \tilde{e}_g (also \tilde{w}) is still finely and cofinely continuous with quasi every starting point of \mathbf{M} , and except on an exceptional set N , \tilde{e}_g (also \tilde{w}) is characterized as a version of the solution of a variational inequality problem.

Recently, Palczewski and Stettner [9][10] used the penalty method to characterize the continuity of the value function of a time inhomogeneous optimal stopping problem. In their work, the underlying process \mathbf{M} is assumed to satisfy the Feller continuity property. Lamberton [4] derived the continuity property of the value function of a one-dimensional optimal stopping problem, and the value function was characterized as the unique solution of a variational inequality in the sense of distributions. However, that result was difficult to be extended to multi-dimensional diffusions.

In this paper, through the time dependent Dirichlet form, it is showed that under the absolute continuity condition on the transition probability function p_t and some other assumptions, Oshima's [8] results still hold and there does not exist the exceptional set for the starting points of \mathbf{M} . This result is then applied in Section 4 to the time inhomogeneous optimal stopping games where the underlying process is a multi-dimensional Ito diffusion.

2 Time Dependent Dirichlet Form

In this section we define the settings for the time dependent Dirichlet form that are similar to those in [8], although some results from [11], whose notions are different, will be used later. Let \mathbb{X} be a locally compact separable metric space and \mathbf{m} be a positive Radon measure on \mathbb{X} with full support. For each $t \geq 0$, define $(E^{(t)}, F)$ as an \mathbf{m} -symmetric Dirichlet form on $H = L^2(\mathbb{X}; \mathbf{m})$ and for any $u \in F$, we assume that $E^{(t)}(u, u)$ is a measurable function of t and satisfies

$$\lambda^{-1}\|u\|_F^2 \leq E_1^{(t)}(u, u) \leq \lambda\|u\|_F^2$$

for some constant $\lambda > 0$, where $E_\alpha^{(t)}(u, v) = E^{(t)}(u, v) + \alpha(u, v)_{\mathbf{m}}$, $\alpha > 0$, and the F norm is defined to be $\|u\|_F^2 = E_1^{(0)}(u, u)$. We also assume that F is regular and local in the usual sense [1].

Define F' as the dual space of F , then it can be seen that $F \subset H = H' \subset F'$. For each t , there exists an operator $L^{(t)}$ from F to F' such that

$$-(L^{(t)}u, v) = E^{(t)}(u, v), \quad u, v \in F.$$

Further the F' norm is defined as

$$\|v\|_{F'} = \sup_{\|u\|_F=1} \{(v, u)\},$$

where (v, u) denotes the canonical coupling between $v \in F'$ and $u \in F$.

Define the spaces

$$\mathcal{H} = \{\varphi(t, \cdot) \in H : \|\varphi\|_{\mathcal{H}} < \infty\},$$

where

$$\|\varphi\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \|\varphi(t, \cdot)\|_H^2 dt,$$

and

$$\mathcal{F} = \{\varphi(t, x) \in F : \|\varphi\|_{\mathcal{F}} < \infty\},$$

where

$$\|\varphi\|_{\mathcal{F}}^2 = \int_{\mathbb{R}} \|\varphi(t, \cdot)\|_F^2 dt.$$

Clearly $\mathcal{F} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{F}'$ densely and continuously, where \mathcal{H}' , \mathcal{F}' are the dual spaces of \mathcal{H} , \mathcal{F} respectively.

For any $\varphi \in \mathcal{F}$, considering φ as a function of $t \in \mathbb{R}$ with values in F , the distribution derivative $\partial\varphi/\partial t$ is considered as a function of $t \in \mathbb{R}$ with values in F' such that

$$\int_{\mathbb{R}} \frac{\partial\varphi}{\partial t}(t, \cdot) \xi(t) dt = - \int_{\mathbb{R}} \varphi(t, \cdot) \xi'(t) dt,$$

for any $\xi \in C_0^\infty(\mathbb{R})$. Then we can define the space \mathcal{W} as

$$\mathcal{W} = \{\varphi(t, x) \in \mathcal{F} : \frac{\partial\varphi}{\partial t} \in \mathcal{F}', \|\varphi\|_{\mathcal{W}} < \infty\},$$

where

$$\|\varphi\|_{\mathcal{W}}^2 = \left\| \frac{\partial\varphi}{\partial t} \right\|_{\mathcal{F}'}^2 + \|\varphi\|_{\mathcal{F}}^2.$$

Since \mathcal{F} and \mathcal{F}' are Banach spaces, it is easy to see that \mathcal{W} is also a Banach space. Further, \mathcal{W} is dense in \mathcal{F} .

We further define the bilinear form \mathcal{E} by

$$\mathcal{E}(\varphi, \psi) = \begin{cases} -\langle \frac{\partial\varphi}{\partial t}, \psi \rangle + \int_{\mathbb{R}} E^{(t)}(\varphi(t, \cdot), \psi(t, \cdot)) dt, & \varphi \in \mathcal{W}, \psi \in \mathcal{F}, \\ \langle \frac{\partial\psi}{\partial t}, \varphi \rangle + \int_{\mathbb{R}} E^{(t)}(\varphi(t, \cdot), \psi(t, \cdot)) dt, & \varphi \in \mathcal{F}, \psi \in \mathcal{W}, \end{cases} \quad (3)$$

where $\langle \frac{\partial\varphi}{\partial t}, \psi \rangle = \int_{\mathbb{R}} (\frac{\partial\varphi}{\partial t}, \psi) dt$. We call $(\mathcal{E}, \mathcal{F})$ a time dependent Dirichlet form on \mathcal{H} [8].

As in [8] we may introduce the time space process $Z_t = (\tau(t), X_t)$ on the domain $\mathbb{Z} = \mathbb{R} \times \mathbb{X}$ with uniform motion $\tau(t)$, then the resolvent $R_\alpha f$ of Z_t defined by

$$R_\alpha f(s, x) = E_{(s, x)} \left(\int_0^\infty e^{-\alpha t} f(s + t, X_{s+t}) dt \right), \quad (s, x) = z, \quad f \in \mathcal{H}, \quad (4)$$

satisfies

$$\left(\alpha - \frac{\partial}{\partial t} - L^{(t)} \right) R_\alpha f(t, x) = f(t, x), \quad \forall t \geq 0. \quad (5)$$

Furthermore, $R_\alpha f$ is considered as a version of $G_\alpha f \in \mathcal{W}$, where G_α is the resolvent associated with the form $\mathcal{E}_\alpha(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \alpha(\cdot, \cdot)_\nu$ and it satisfies

$$\mathcal{E}_\alpha(G_\alpha f, \varphi) = (f, \varphi)_\nu, \quad \forall \varphi \in \mathcal{F}, \quad (6)$$

where $d\nu(t, x) = dt d\mathbf{m}(x)$. We may write $(\cdot, \cdot)_\nu$ as $(\cdot, \cdot)_\mathcal{H}$ to indicate it as the inner product in \mathcal{H} .

We now define \mathcal{A} as a bilinear form on $\mathcal{F} \times \mathcal{F}$ by

$$\mathcal{A}(\varphi, \psi) = \int_{\mathbb{R}} E^{(t)}(\varphi(t, \cdot), \psi(t, \cdot)) dt,$$

and set $\mathcal{A}_\alpha(\varphi, \psi) = \mathcal{A}(\varphi, \psi) + \alpha(\varphi, \psi)_\mathcal{H}$, then it can be seen that $\mathcal{E}_\alpha(\varphi, \psi) = -\langle \frac{\partial \varphi}{\partial t}, \psi \rangle + \mathcal{A}_\alpha(\varphi, \psi)$ if $\varphi \in \mathcal{W}$, $\psi \in \mathcal{F}$, and $\mathcal{E}_\alpha(\varphi, \psi) = \langle \frac{\partial \psi}{\partial t}, \varphi \rangle + \mathcal{A}_\alpha(\varphi, \psi)$ if $\varphi \in \mathcal{F}$, $\psi \in \mathcal{W}$. Also notice that if $\varphi \in \mathcal{W}$, $\langle \frac{\partial \varphi}{\partial t}, \varphi \rangle = 0$, hence $\mathcal{E}_\alpha(\varphi, \varphi) = \mathcal{A}_\alpha(\varphi, \varphi)$ in this case, see Corollary 1.1 in [7].

A function $\varphi \in \mathcal{F}$ is called α -potential if $\mathcal{E}_\alpha(\varphi, \psi) \geq 0$ for any nonnegative function $\psi \in \mathcal{W}$. Denote by \mathcal{P}_α the family of all α -potential functions. A function $\varphi \in \mathcal{F}$ is called α -excessive if and only if $\varphi \geq 0$ and $nG_{n+\alpha}\varphi \leq \varphi$ a.e. for any $n \geq 0$. For any α -potential $\varphi \in \mathcal{F}$, define its α -excessive modification as

$$\tilde{\varphi} = \lim_{n \rightarrow \infty} nR_{n+\alpha}\varphi.$$

For any function $g \in \mathcal{H}$, let

$$\mathcal{L}_g = \{\varphi \in \mathcal{F} : \varphi \geq g \text{ } \nu \text{ a.e.}\},$$

then the following result holds (see Lemma 1.1 in [8]):

Lemma 2.1. *For any $\epsilon > 0$ and $\alpha > 0$, there exists a unique function $g_\epsilon^\alpha \in \mathcal{W}$ such that*

$$-\left(\frac{\partial g_\epsilon^\alpha}{\partial t}, u \right) + E_\alpha^{(t)}(g_\epsilon^\alpha(t, \cdot), u) = \frac{1}{\epsilon} ((g_\epsilon^\alpha(t, \cdot) - g(t, \cdot))^- , u) \quad (7)$$

for any $u \in F$.

As a consequence, g_ϵ^α solves

$$\mathcal{E}_\alpha(g_\epsilon^\alpha, \psi) = \frac{1}{\epsilon}((g_\epsilon^\alpha - g)^-, \psi)_{\mathcal{H}}, \quad \forall \psi \in \mathcal{F}, \quad (8)$$

see Proposition 1.6 in [11].

By Theorem 1.2 in [8], $e_g = \lim_{\epsilon \rightarrow 0} g_\epsilon^\alpha$ converges increasingly, strongly in \mathcal{H} and weakly in \mathcal{F} , and furthermore, e_g is the minimal function of $\mathcal{P}_\alpha \cap \mathcal{L}_g$ satisfying

$$\mathcal{A}_\alpha(e_g, e_g) \leq \mathcal{E}_\alpha(e_g, \psi), \quad \forall \psi \in \mathcal{L}_g \cap \mathcal{W}. \quad (9)$$

Given any open set $A \in \mathbb{Z}$, the capacity of A is defined by

$$\text{Cap}(A) = \mathcal{E}_\alpha(e_{I_A}, \psi), \quad \psi \in \mathcal{W}, \psi = 1 \text{ a.e. on } A.$$

If $\varphi \in \mathcal{F}$ is an α -potential, then there exists a positive Radon measure μ_φ^α on \mathbb{Z} such that

$$\mathcal{E}_\alpha(\varphi, \psi) = \int_{\mathbb{Z}} \psi(z) d\mu_\varphi^\alpha(z) \quad \text{for any } \psi \in C_0(\mathbb{Z}) \cap \mathcal{W}$$

By Lemma 1.4 in [8], μ_φ^α does not charge any Borel set of zero capacity. Put $e_A = e_{I_A}$ and $\mu_A^\alpha = \mu_{e_A}^\alpha$, then the capacity of the set A can also be defined by

$$\text{Cap}(A) = \mu_A^\alpha(\bar{A}).$$

The notion of the capacity is extended to any Borel set by the usual manner. A set is called exceptional if it is of zero capacity. If a statement holds everywhere except on an exceptional set, we say the statement holds quasi-everywhere (q.e.).

3 The Time Inhomogeneous Stopping Games

In this section we will characterize the properties of the value functions $\tilde{e}_g = \sup_\sigma J_{(s,x)}(\sigma)$ and $\tilde{w} = \sup_\sigma \inf_\tau J_{(s,x)}(\tau, \sigma)$ of the time inhomogeneous stopping games. We first assume that the transition probability function p_t of the process X_t satisfies the *absolute continuity* condition:

$$p_t(x, \cdot) \ll m(\cdot), \quad \forall t. \quad (10)$$

In fact, the Feller property in [9] implies the absolute continuity condition on p_t , see, e.g., page 165 of [1].

3.1 The Time Inhomogeneous Optimal Stopping Problem

Consider $\tilde{e}_g(z) = \sup_\sigma J_z(\sigma)$ where

$$J_z(\sigma) = J_{(s,x)}(\sigma) = E_{(s,x)}(e^{-\alpha\sigma} g(s + \sigma, X_{s+\sigma})). \quad (11)$$

Oshima showed that (see Theorem 3.1 in [8]) if $g \in \mathcal{F}$ is quasi continuous and $\mathcal{L}_g \cap \mathcal{W} \neq \emptyset$, then $\tilde{e}_g(z) \in \mathcal{F}$ is finely and cofinely continuous q.e., and e_g solves the variational inequality (9). In what follows we give conditions under which $\tilde{e}_g \in \mathcal{W}$ and Oshima's result holds without the exceptional set.

It is assumed that $g \in \mathcal{W}$ is a finely continuous function on \mathbb{Z} such that

$$|g(t, x)| \leq \varphi(t, x), \quad (12)$$

for some finite α -excessive function $\varphi \in \mathcal{W}$ on \mathbb{Z} . We also assume that there exists a constant K such that

$$\sup_{\epsilon > 0} \frac{1}{\epsilon} \|(g_\epsilon^\alpha - g)^-\|_{\mathcal{H}} \leq K \|g\|_{\mathcal{H}}, \quad (13)$$

where g_ϵ^α solves (7). In the rest of this section, the notion K_i for some index i denotes a constant.

Lemma 3.1. *Under the assumptions (12) and (13), $e_g \in \mathcal{W}$.*

Proof. It has been proved that $e_g \in \mathcal{L}_g \cap P_\alpha$, and $e_g \in \mathcal{F}$, see Theorem 1.2 of [8] or Proposition 1.7 of [11]. Furthermore,

$$\sup_{\epsilon} \|g_\epsilon^\alpha - \varphi\|_{\mathcal{F}} \leq K_1 \|\varphi\|_{\mathcal{W}},$$

and

$$\sup_{\epsilon} \|g_\epsilon^\alpha\|_{\mathcal{F}} \leq \sup_{\epsilon} \|g_\epsilon^\alpha - \varphi\|_{\mathcal{F}} + \|\varphi\|_{\mathcal{F}} \leq K_1 \|\varphi\|_{\mathcal{W}} + \|\varphi\|_{\mathcal{F}}.$$

Now that g_ϵ^α satisfies

$$\left\langle -\frac{\partial g_\epsilon^\alpha}{\partial t}, \psi \right\rangle + \mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) = \frac{1}{\epsilon} \left((g_\epsilon^\alpha - g)^-, \psi \right)_{\mathcal{H}}, \quad \forall \psi \in \mathcal{F},$$

we have

$$\begin{aligned} \left\| \frac{\partial g_\epsilon^\alpha}{\partial t} \right\|_{\mathcal{F}'} &= \sup_{\|\psi\|_{\mathcal{F}}=1} \left\langle \frac{\partial g_\epsilon^\alpha}{\partial t}, \psi \right\rangle \\ &= \sup_{\|\psi\|_{\mathcal{F}}=1} \left(\mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) - \frac{1}{\epsilon} \left((g_\epsilon^\alpha - g)^-, \psi \right)_{\mathcal{H}} \right) \\ &\leq \sup_{\|\psi\|_{\mathcal{F}}=1} \mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) + \sup_{\|\psi\|_{\mathcal{F}}=1} \frac{1}{\epsilon} \left((g_\epsilon^\alpha - g)^-, \psi \right)_{\mathcal{H}}. \end{aligned} \quad (14)$$

By the sector condition,

$$\mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) \leq K_2 \|g_\epsilon^\alpha\|_{\mathcal{F}} \|\psi\|_{\mathcal{F}},$$

hence

$$\sup_{\|\psi\|_{\mathcal{F}}=1} \mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) \leq K_2 \|g_\epsilon^\alpha\|_{\mathcal{F}}.$$

On the other hand, by Cauchy-Schwarz inequality, the following holds:

$$\frac{1}{\epsilon} \left((g_\epsilon^\alpha - g)^-, \psi \right)_{\mathcal{H}} \leq \frac{1}{\epsilon} \|g_\epsilon^\alpha - g\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}},$$

and

$$\|\psi\|_{\mathcal{H}} \leq K_3 \|\psi\|_{\mathcal{F}},$$

hence

$$\sup_{\|\psi\|_{\mathcal{F}}=1} \frac{1}{\epsilon} \left((g_\epsilon^\alpha - g)^-, \psi \right)_{\mathcal{H}} \leq \frac{1}{\epsilon} K_3 \|g_\epsilon^\alpha - g\|_{\mathcal{H}}.$$

Now by taking \sup_ϵ of (14) and by (13) we get

$$\sup_\epsilon \left\| \frac{\partial g_\epsilon^\alpha}{\partial t} \right\|_{\mathcal{F}'} \leq K_2 K_1 \|\varphi\|_{\mathcal{W}} + K_2 \|\varphi\|_{\mathcal{F}} + K K_3 \|g\|_{\mathcal{H}} < \infty.$$

Therefore

$$\begin{aligned} \sup_\epsilon \|g_\epsilon^\alpha\|_{\mathcal{W}} &= \sup_\epsilon \left(\left\| \frac{\partial g_\epsilon^\alpha}{\partial t} \right\|_{\mathcal{F}'} + \|g_\epsilon^\alpha\|_{\mathcal{F}} \right) \\ &\leq K_2 K_1 \|\varphi\|_{\mathcal{W}} + K_2 \|\varphi\|_{\mathcal{F}} + K K_3 \|g\|_{\mathcal{H}} + K_1 \|\varphi\|_{\mathcal{W}} + \|\varphi\|_{\mathcal{F}}, \end{aligned} \quad (15)$$

and as a consequence, $e_g \in \mathcal{W}$ by Lemma I.2.12 in [5]. \square

Corollary 3.1. *There exist constants K_4, K_5 such that $\|e_g\|_{\mathcal{W}} \leq K_4 \|\varphi\|_{\mathcal{W}} + K_5 \|g\|_{\mathcal{H}}$.*

Proof. This can be seen by Eq.(15) in the proof of Lemma 3.1 and the fact that $\|\varphi\|_{\mathcal{F}} \leq \|\varphi\|_{\mathcal{W}}$. \square

Now we can revise Theorem 1.2 of [8] and get the following result.

Corollary 3.2. *Under the assumptions (12) and (13), $e_g = \lim_{\epsilon \rightarrow 0} g_\epsilon^\alpha$ converges increasingly, strongly in \mathcal{H} , and weakly in both \mathcal{F} and \mathcal{W} . Furthermore, e_g is the minimal function of $\mathcal{P}_\alpha \cap \mathcal{L}_g \cap \mathcal{W}$ satisfying*

$$\mathcal{E}_\alpha(e_g, e_g) \leq \mathcal{E}_\alpha(e_g, \psi), \quad \forall \psi \in \mathcal{L}_g \cap \mathcal{W}. \quad (16)$$

Proof. Now that $e_g \in \mathcal{W}$, so $\langle \frac{\partial e_g}{\partial t}, e_g \rangle = 0$ (see Lemma 1.1 of [7]), hence $\mathcal{A}_\alpha(e_g, e_g) = \mathcal{E}_\alpha(e_g, e_g)$. The rest of the proof is the same as in [8]. \square

Theorem 3.1. *Let $g(z) = g(t, x)$ be a finely continuous function satisfying (12). Assume (13) and the absolute continuity condition (10). Let e_g be the solution of (16), and \tilde{e}_g be its α -excessive regularization. Then*

$$\tilde{e}_g(z) = \sup_\sigma J_z(\sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (17)$$

where $J_z(\sigma) = J_{(s,x)}(\sigma) = E_{(s,x)}(e^{-\alpha\sigma} g(s + \sigma, X_{s+\sigma}))$. Furthermore, let the set $B = \{z \in \mathbb{Z} : \tilde{e}_g(z) = g(z)\}$ and let σ_B be the first hitting time of B defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_g(Z_{s+t}) = g(Z_{s+t})\}$, then

$$\tilde{e}_g(z) = E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})]. \quad (18)$$

Proof. Notice that $\varphi \wedge \tilde{e}_g$ is an α -potential dominating g , and \tilde{e}_g is the smallest α -potential dominating g , we get $\tilde{e}_g \leq \varphi \wedge \tilde{e}_g \leq \varphi$ ν -a.e., which implies the finiteness of \tilde{e}_g .

Now because $e_g \geq g$ ν a.e., we have $nR_{n+\alpha}e_g(z) \geq nR_{n+\alpha}g(z)$, $\forall z \in \mathbb{Z}$, $n > 0$, and this implies

$$\tilde{e}_g(z) \geq \lim_{n \rightarrow \infty} nR_{n+\alpha}g(z), \quad \forall z \in \mathbb{Z}.$$

By the absolute continuity condition and applying the dominated convergence theorem, the following holds,

$$\lim_{n \rightarrow \infty} nR_{n+\alpha}g(z) = g(z), \quad \forall z \in \mathbb{Z},$$

therefore $\tilde{e}_g(z) \geq g(z)$, $\forall z \in \mathbb{Z}$. Then we have

$$\tilde{e}_g(z) \geq E_z(e^{-\alpha\sigma}\tilde{e}_g(Z_{s+\sigma})) \geq E_z(e^{-\alpha\sigma}g(Z_{s+\sigma})), \quad (19)$$

for any stopping time σ , which implies $\tilde{e}_g(z) \geq J_z(\sigma)$, $\forall z \in \mathbb{Z}$, hence $\tilde{e}_g(z) \geq \sup_{\sigma} J_z(\sigma)$, $\forall z \in \mathbb{Z}$.

Since e_g is a bounded α -potential, there exists a positive Radon measure μ^α of finite energy such that

$$\mathcal{E}_\alpha(e_g, w) = \int_{\mathbb{Z}} w(z) \mu^\alpha(dz), \quad \forall w \in C_0(\mathbb{Z}) \cap \mathcal{W}, \quad (20)$$

and $\tilde{e}_g(z) = R_\alpha \mu(z)$.

Under the absolute continuity condition (10) of the transition function, there exists a positive continuous additive functional A_t in the strict sense (see Theorem 5.1.6 in [1]) such that

$$\tilde{e}_g(z) = E_z \left(\int_0^\infty e^{-\alpha t} dA_t \right), \quad \forall z \in \mathbb{Z}.$$

Set $B = \{z \in \mathbb{Z} : \tilde{e}_g(z) = g(z)\}$, then

$$\int_{B^c} (\tilde{e}_g(z) - g(z)) \mu^\alpha(dz) = \int_{\mathbb{Z}} (\tilde{e}_g(z) - g(z)) \mu^\alpha(dz) = \mathcal{E}_\alpha(e_g, e_g - g).$$

Since e_g is an α -potential, and $e_g - g$ is nonnegative, $\mathcal{E}_\alpha(e_g, e_g - g) \geq 0$, which implies $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) \geq 0$. On the other hand, e_g satisfies (16), which implies $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) \leq 0$. Now it can be concluded that $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) = 0$, hence $\mu^\alpha(B^c) = 0$. Further we get

$$E_z \left(\int_0^\infty e^{-\alpha t} I_{B^c}(Z_{s+t}) dA_t \right) = R_\alpha(I_{B^c}\mu)(z) = 0, \quad \forall z \in \mathbb{Z}.$$

By the strong Markov property, we have for any stopping time $\sigma \leq \sigma_B$

$$\tilde{e}_g(z) = E_z \left[\int_0^\sigma e^{-\alpha t} dA_t \right] + E_z[e^{-\alpha\sigma}\tilde{e}_g(Z_{s+\sigma})], \quad (21)$$

and because

$$0 \leq E_z \left[\int_0^\sigma e^{-\alpha t} dA_t \right] \leq E_z \left(\int_0^\infty e^{-\alpha t} I_{B^c}(Z_{s+t}) dA_t \right) = 0,$$

we have $\tilde{e}_g(z) = E_z[e^{-\alpha\sigma}\tilde{e}_g(Z_{s+\sigma})]$, $\sigma \leq \sigma_B$. By replacing σ by σ_B and replacing $\tilde{e}_g(Z_{s+\sigma})$ by $g(Z_{s+\sigma_B})$, we get $\tilde{e}_g(z) = E_z[e^{-\alpha\sigma_B}g(Z_{s+\sigma_B})]$, and this together with (19) completes the proof. \square

Corollary 3.3. *Under the conditions in Theorem 3.1, $\tilde{e}_g(z)$ is finely and cofinely continuous for all $z \in \mathbb{Z}$.*

Proof. Oshima [8] has showed that $\tilde{e}_g(z)$ is finely and cofinely continuous for q.e. z , and under the conditions in Theorem 3.1, we showed that there does not exist the exceptional set, so $\tilde{e}_g(z)$ is finely and cofinely continuous for all $z \in \mathbb{Z}$. \square

Remark 3.1. *Since X_t is a diffusion process, we can see that $\tilde{e}_g(z)$ is continuous along the sample paths.*

In Palczewski and Stettner's work [9][10], the optimal policy is to stop the game at the stopping time $\dot{\sigma}_B = \inf\{t \geq 0 : \tilde{e}_g(Z_{s+t}) \leq g(Z_{s+t})\}$ or equivalently $\dot{\sigma}_B = \inf\{t \geq 0 : \tilde{e}_g(Z_{s+t}) = g(Z_{s+t})\}$. Notice that $\dot{\sigma}_B \leq \sigma_B$, by Theorem 3.1 we can see that

$$\tilde{e}_g(z) \geq E_z[e^{-\alpha\dot{\sigma}_B}\tilde{e}_g(Z_{s+\dot{\sigma}_B})] \geq E_z[e^{-\alpha\sigma_B}\tilde{e}_g(Z_{s+\sigma_B})] \geq E_z[e^{-\alpha\sigma_B}g(Z_{s+\sigma_B})] = \tilde{e}_g(z), \quad \forall z,$$

hence

$$E_z[e^{-\alpha\dot{\sigma}_B}\tilde{e}_g(Z_{s+\dot{\sigma}_B})] = E_z[e^{-\alpha\sigma_B}\tilde{e}_g(Z_{s+\sigma_B})], \quad \forall z,$$

and as a byproduct, we get the following result:

Corollary 3.4. *Under the conditions in Theorem 3.1, there does not exist the exceptional set of irregular boundary points of B .*

Therefore it is feasible to replace σ_B by $\dot{\sigma}_B$ in the results in the rest of this paper.

3.2 The Time Inhomogeneous Zero-sum Game

In this section we will refine the solution of the two-obstacle problem (zero-sum game) in [8].

Let $g(t, x), h(t, x) \in \mathscr{W}$ be finely continuous functions satisfying

$$g(t, x) \leq h(t, x), \quad |g(t, x)| \leq \varphi(t, x), \quad |h(t, x)| \leq \psi(t, x), \quad \forall (t, x) \in \mathbb{Z}, \quad (22)$$

where $\varphi, \psi \in \mathscr{W}$ are two bounded α -excessive functions. We also assume that g, h satisfy the condition (13). Suppose there exist bounded α -excessive functions $v_1(t, x), v_2(t, x) \in \mathscr{W}$ such that

$$g(t, x) \leq v_1(t, x) - v_2(t, x) \leq h(t, x), \quad \forall (t, x) \in \mathbb{Z}, \quad (23)$$

in which case we say g and h satisfy the *separability condition* [3].

Define the sequences of α -excessive functions $\{\varphi_n\}$ and $\{\psi_n\}$ inductively by

$$\varphi_0 = \psi_0 = 0, \psi_n = e_{\varphi_{n-1}-h}, \varphi_n = e_{\psi_n+g}, \quad n \geq 1,$$

then the following holds:

Lemma 3.2. *Assuming (23), then φ_n, ψ_n are well defined and $\lim_{n \rightarrow \infty} \varphi_n = \bar{\varphi}$, $\lim_{n \rightarrow \infty} \psi_n = \bar{\psi}$ converge increasingly, strongly in \mathcal{H} and weakly in both \mathcal{F} and \mathcal{W} .*

Proof. We only need to show the convergence in \mathcal{W} and the rest of this lemma is just Lemma 2.1 in [8]. Firstly $\varphi_0 = 0 \leq v_1$ and $\varphi_0 \in \mathcal{W}$. Suppose $\varphi_{n-1} \in \mathcal{W}$ is well defined and satisfies $\varphi_{n-1} \leq v_1$, then $\varphi_{n-1} - h \leq v_1 - h \leq v_2$. Hence $\psi_n = e_{\varphi_{n-1}-h} \in \mathcal{W}$ is well defined by Lemma 3.1, and we also have $\psi_n \leq v_2$ since $e_{\varphi_{n-1}-h}$ is the smallest α -potential dominating $\varphi_{n-1} - h$. Now that $\psi_n + g \leq v_2 + g \leq v_1$, hence $\varphi_n = e_{\psi_n+g} \in \mathcal{W}$ is well defined and is dominated by v_1 .

Notice that $\varphi_0 \leq \varphi_1$. Suppose $\varphi_{n-1} \leq \varphi_n$, then $\psi_n = e_{\varphi_{n-1}-h} \leq e_{\varphi_n-h} = \psi_{n+1}$, hence $\varphi_n = e_{\psi_n+g} \leq e_{\psi_{n+1}+g} = \varphi_{n+1}$. Also by Lemma 3.1 we get

$$\|\varphi_n\|_{\mathcal{W}} = \|e_{\psi_n+g}\|_{\mathcal{W}} \leq K_4 \|v_1\|_{\mathcal{W}} + K_5 \|\psi_n + g\|_{\mathcal{H}}.$$

Notice that $g \leq \psi_n + g \leq v_1$, hence $\|\psi_n + g\|_{\mathcal{H}}$ is uniformly bounded in n , and as a consequence, $\|\varphi_n\|_{\mathcal{W}}$ is uniformly bounded in n . In a similar manner we can show that $\|\psi_n\|_{\mathcal{W}}$ is uniformly bounded. The convergence of φ_n, ψ_n in \mathcal{W} follows by Lemma I.2.12 in [5]. \square

Corollary 3.5. *Under the separability condition, $\bar{\varphi} = e_{\bar{\psi}+g}$, $\bar{\psi} = e_{\bar{\varphi}-h}$, and they satisfy*

$$\begin{aligned} \mathcal{E}_\alpha(\bar{\varphi}, \bar{\varphi}) &\leq \mathcal{E}_\alpha(\bar{\varphi}, w), \quad \forall w \in \mathcal{L}_{\bar{\psi}+g} \cap \mathcal{W}, \\ \mathcal{E}_\alpha(\bar{\psi}, \bar{\psi}) &\leq \mathcal{E}_\alpha(\bar{\psi}, w), \quad \forall w \in \mathcal{L}_{\bar{\varphi}-h} \cap \mathcal{W}. \end{aligned} \quad (24)$$

Proof. Since $\bar{\varphi}$ is an α -potential dominating $\bar{\psi} + g$, we get $e_{\bar{\psi}+g} \leq \bar{\varphi}$. On the other hand, $\bar{\varphi} = \lim_{n \rightarrow \infty} \varphi_n = \lim_{n \rightarrow \infty} e_{\psi_n+g} \leq e_{\bar{\psi}+g}$, hence $\bar{\varphi} = e_{\bar{\psi}+g}$. Similarly $\bar{\psi} = e_{\bar{\varphi}-h}$. The proof of (24) is immediate by Corollary 3.2. \square

Corollary 3.6. *If a pair of α -excessive functions (V_1, V_2) satisfy $g \leq V_1 - V_2 \leq h$, then $\bar{\varphi} \leq V_1$, $\bar{\psi} \leq V_2$, and $\bar{w} := \bar{\varphi} - \bar{\psi}$ is the unique function in \mathcal{J} satisfying*

$$\mathcal{E}_\alpha(\bar{w}, \bar{w}) \leq \mathcal{E}_\alpha(\bar{w}, w), \quad \forall w \in \mathcal{J}, \quad g \leq w \leq h, \quad (25)$$

where $\mathcal{J} = \{w = \varphi_1 - \varphi_2 + v : \varphi_1, \varphi_2 \in \mathcal{W} \text{ are } \alpha\text{-potentials}, v \in \mathcal{W}\}$.

Proof. Clearly $\varphi_{n-1} - h \leq \psi_n$ and $\psi_n + g \leq \bar{\varphi}_n$, hence $g \leq \bar{\varphi} - \bar{\psi} \leq h$. If g, h satisfy the separability condition with respect to V_1, V_2 , then we would have $\varphi_n \leq V_1$ and $\psi_n \leq V_2$, and as a consequence $\bar{\varphi} \leq V_1$, $\bar{\psi} \leq V_2$.

Now (25) is equivalent to

$$\mathcal{E}_\alpha(\bar{\varphi}, \bar{\varphi}) + \mathcal{E}_\alpha(\bar{\psi}, \bar{\psi}) \leq \mathcal{E}_\alpha(\bar{\varphi}, w + \bar{\psi}) + \mathcal{E}_\alpha(\bar{\psi}, \bar{\varphi} - w), \quad g \leq w \leq h,$$

which holds by (24). Suppose there are two solutions $\bar{w}_1, \bar{w}_2 \in \mathcal{J}$ satisfying (25). Notice that $\bar{w}_1 - \bar{w}_2 \in \mathcal{W}$ and $\langle \frac{\partial(\bar{w}_1 - \bar{w}_2)}{\partial t}, \bar{w}_1 - \bar{w}_2 \rangle = 0$, so

$$\langle \frac{\partial \bar{w}_1}{\partial t}, \bar{w}_2 \rangle + \langle \frac{\partial \bar{w}_2}{\partial t}, \bar{w}_1 \rangle = 0,$$

and consequently

$$\mathcal{A}_\alpha(\bar{w}_1, \bar{w}_2) + \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_1) = \mathcal{E}_\alpha(\bar{w}_1, \bar{w}_2) + \mathcal{E}_\alpha(\bar{w}_2, \bar{w}_1).$$

Therefore,

$$\begin{aligned} \mathcal{A}_\alpha(\bar{w}_1 - \bar{w}_2, \bar{w}_1 - \bar{w}_2) &= \mathcal{A}_\alpha(\bar{w}_1, \bar{w}_1) + \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_2) - \mathcal{A}_\alpha(\bar{w}_1, \bar{w}_2) - \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_1) \\ &= \mathcal{A}_\alpha(\bar{w}_1, \bar{w}_1) + \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_2) - \mathcal{E}_\alpha(\bar{w}_1, \bar{w}_2) - \mathcal{E}_\alpha(\bar{w}_2, \bar{w}_1) \leq 0, \end{aligned}$$

which implies that $\bar{w}_1 = \bar{w}_2$ a.e. \square

Let $\tilde{\varphi}, \tilde{\psi}, \tilde{w}$ be the α -excessive modifications of $\bar{\varphi}, \bar{\psi}, \bar{w}$, respectively. We further define for arbitrary pair of stopping times τ, σ the payoff function $J_z(\tau, \sigma)$ as

$$J_z(\tau, \sigma) = E_z \left[e^{-\alpha(\tau \wedge \sigma)} (g(Z_{s+\sigma}) I_{\tau > \sigma} + h(Z_{s+\tau}) I_{\tau \leq \sigma}) \right], \quad z \in \mathbb{Z}. \quad (26)$$

Then we have the following result:

Theorem 3.2. *Assuming the separability condition on g, h and conditions (10) (13). There exists a finite finely and cofinely continuous function $\tilde{w}(z) \in \mathcal{J}$ satisfying (25) and the identity*

$$\tilde{w}(z) = \sup_{\sigma} \inf_{\tau} J_z(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_z(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (27)$$

where σ, τ range over all stopping times. Moreover, the pair $\hat{\tau}, \hat{\sigma}$ defined by

$$\hat{\tau} = \inf\{t > 0 : \bar{w}(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}(Z_{s+t}) = g(Z_{s+t})\}$$

is the saddle point of the game in the sense that

$$J_z(\hat{\tau}, \sigma) \leq J_z(\hat{\tau}, \hat{\sigma}) \leq J_z(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},$$

for all stopping times τ, σ .

Proof. We only need to prove (27). By Theorem 3.1, for any $z \in \mathbb{Z}$ we have

$$\begin{aligned} \tilde{\varphi}(z) &= \sup_{\sigma} E_z[e^{-\alpha\sigma}(\tilde{\psi} + g)(Z_{s+\sigma})] = E_z[e^{-\alpha\hat{\sigma}}(\tilde{\psi} + g)(Z_{s+\hat{\sigma}})], \\ \tilde{\psi}(z) &= \sup_{\tau} E_z[e^{-\alpha\tau}(\tilde{\varphi} - h)(Z_{s+\tau})] = E_z[e^{-\alpha\hat{\tau}}(\tilde{\varphi} - h)(Z_{s+\hat{\tau}})], \end{aligned} \quad (28)$$

and for any stopping times $\sigma \leq \hat{\sigma}, \tau \leq \hat{\tau}$,

$$\tilde{\varphi}(z) = E_z[e^{-\alpha\sigma}\tilde{\varphi}(Z_{s+\sigma})], \quad \tilde{\psi}(z) = E_z[e^{-\alpha\tau}\tilde{\psi}(Z_{s+\tau})], \quad \forall z = (s, x) \in \mathbb{Z}.$$

From (21), we could take $\{e^{-\alpha t}\bar{\varphi}(Z_{s+t})\}$ and $\{e^{-\alpha t}\bar{v}(Z_{s+t})\}$ as non-negative P_z -supermartingales, therefore, for any $z \in \mathbb{Z}$ and any stopping times τ, σ , we have

$$\tilde{\varphi}(z) \geq E_z[e^{-\alpha\sigma}\tilde{\varphi}(Z_{s+\sigma})], \quad \tilde{\psi}(z) \geq E_z[e^{-\alpha\tau}\tilde{\psi}(Z_{s+\tau})].$$

Consequently, for any $z \in \mathbb{Z}$,

$$\begin{aligned} \tilde{w}(z) &= \tilde{\varphi}(z) - \tilde{\psi}(z) \leq E_z[e^{-\alpha(\hat{\sigma} \wedge \tau)}\tilde{\varphi}(Z_{s+\hat{\sigma} \wedge \tau})] - E_z[e^{-\alpha(\hat{\sigma} \wedge \tau)}\tilde{\psi}(Z_{s+\hat{\sigma} \wedge \tau})] \\ &= E_z[e^{-\alpha(\hat{\sigma} \wedge \tau)}\tilde{w}(Z_{s+\hat{\sigma} \wedge \tau})] \leq E_z[e^{-\alpha(\tau \wedge \hat{\sigma})}(g(Z_{s+\sigma}) I_{\tau > \hat{\sigma}} + h(Z_{s+\tau}) I_{\tau \leq \hat{\sigma}})] = J_z(\tau, \hat{\sigma}), \end{aligned}$$

where the last inequality is due to the fact that $g(z) \leq \tilde{w}(z) \leq h(z), \forall z \in \mathbb{Z}$ and (28). In a similar manner, we can prove that $\tilde{w} \geq J_z(\hat{\tau}, \sigma)$, and this completes the proof. \square

3.3 Time Inhomogeneous Stopping Game with Holding Cost

Usually the optimal stopping games involve a holding cost function $f \in \mathcal{H}$, see, e.g., [9], and the return functions become

$$J_{(s,x)}^f(\sigma) = E_{(s,x)} \left(\int_0^\sigma e^{-\alpha t} f(s+t, X_{s+t}) dt + e^{-\alpha \sigma} g(s+\sigma, X_{s+\sigma}) \right), \quad (29)$$

and

$$\begin{aligned} J_{(s,x)}^f(\sigma, \tau) &= E_{(s,x)} \left(\int_0^{\sigma \wedge \tau} e^{-\alpha t} f(s+t, X_{s+t}) dt \right) \\ &\quad + E_{(s,x)} \left(e^{-\alpha(\sigma \wedge \tau)} (g(s+\sigma, X_{s+\sigma}) I_{\sigma < \tau} + h(s+\tau, X_{s+\tau}) I_{\tau \leq \sigma}) \right), \end{aligned} \quad (30)$$

but this model can be essentially reduced to the classical stopping problem by taking $\hat{g} = g - R_\alpha f$ and $\hat{h} = h - R_\alpha f$ instead of g and h respectively, where R_α is the resolvent and $R_\alpha f$ is considered as a version of $G_\alpha f \in \mathcal{W}$. We assume that conditions (12) (13) also apply to \hat{g} for the optimal stopping game (and similarly conditions (22)(23) apply to \hat{g}, \hat{h} for the zero-sum game).

Theorem 3.3. *Let g be a finely continuous function satisfying (12). Assume (13) on g and the absolute continuity condition (10) on p_t . Let e_g^f be the solution of*

$$\mathcal{E}_\alpha(e_g^f, \psi - e_g^f) \geq (f, \psi - e_g^f)_{\mathcal{H}}, \quad \forall \psi \in \mathcal{L}_g \cap \mathcal{W}, \quad (31)$$

and let \tilde{e}_g^f be its α -excessive regularization. Then

$$\tilde{e}_g^f(z) = \sup_{\sigma} J_z^f(\sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (32)$$

where $J_z^f(\sigma)$ is defined as in (29), and $\tilde{e}_g^f(z)$ is finely and cofinely continuous. Furthermore, let the set $B = \{z \in \mathbb{Z} : \tilde{e}_g^f(z) = g(z)\}$ and let σ_B be the first hitting time of B defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = g(Z_{s+t})\}$, then

$$\tilde{e}_g^f(z) = E_z[e^{-\alpha \sigma_B} g(Z_{s+\sigma_B})]. \quad (33)$$

Proof. Define the function

$$J_z^{f0}(\sigma) = E_z(e^{-\alpha \sigma} \hat{g}(s+\sigma, X_{s+\sigma})),$$

where $\hat{g} = g - R_\alpha f$, and let $\tilde{e}_g^f = \sup_{\sigma} J_z^{f0}(\sigma)$, then by Theorem 3.1, e_g^f solves

$$\mathcal{E}_\alpha(e_g^f, \hat{\psi} - e_g^f) \geq 0, \quad \forall \hat{\psi} \in \mathcal{L}_{\hat{g}} \cap \mathcal{W}, \quad (34)$$

and the optimal stopping time is defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = \hat{g}(Z_{s+t})\}$.

By Dynkin's formula,

$$E_{(s,x)} \left(\int_0^\sigma e^{-\alpha t} f(s+t, X_{s+t}) dt \right) = R_\alpha f(s, x) - E_{(s,x)} (e^{-\alpha \sigma} R_\alpha f(s+\sigma, X_{s+\sigma})),$$

which leads to

$$J_z^f(\sigma) = J_z^{f0}(\sigma) + R_\alpha f(z),$$

hence $e_g^f(z) = e_{\hat{g}}^f(z) + R_\alpha f(z)$.

Now let $e_{\hat{g}}^f(z) = e_g^f(z) - R_\alpha f(z)$, $\hat{\psi} = \psi - R_\alpha f$ in (34) we get

$$\mathcal{E}_\alpha(e_g^f - G_\alpha f, \psi - e_g^f) \geq 0. \quad (35)$$

Since $\mathcal{E}_\alpha(G_\alpha f, \psi - e_g^f) = (f, \psi - e_g^f)_\mathcal{H}$, this proves (31). Also notice that the optimal stopping time can be written as $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = g(Z_{s+t})\}$, and this completes the proof. \square

Similarly we can modify Theorem 3.2 and get the following result:

Theorem 3.4. *Let g, h be finely continuous functions satisfying (22) and (23). Assume (13) on g, h and the absolute continuity condition (10) on p_t . Then there exists a finite finely and cofinely continuous function $\tilde{w}^f \in \mathcal{J}$, $g(z) \leq \tilde{w}^f(z) \leq h(z)$, such that*

$$\mathcal{E}_\alpha(\bar{w}^f, w - \bar{w}^f) \geq (f, w - \bar{w}^f)_\mathcal{H}, \quad \forall w \in \mathcal{J}, \quad g \leq w \leq h, \quad (36)$$

and

$$\tilde{w}^f(z) = \sup_{\sigma} \inf_{\tau} J_z^f(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_z^f(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (37)$$

where $J_z^f(\tau, \sigma)$ was given by (30) and σ, τ range over all stopping times. Moreover, the pair $\hat{\tau}, \hat{\sigma}$ defined by

$$\hat{\tau} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = g(Z_{s+t})\}$$

is the saddle point of the game in the sense that

$$J_z^f(\hat{\tau}, \sigma) \leq J_z^f(\hat{\tau}, \hat{\sigma}) \leq J_z^f(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},$$

for all stopping times τ, σ .

As an extension of Corollary 3.6, we have the following:

Corollary 3.7. *The variational inequality (36) has a unique solution.*

Proof. The case where $f = 0$ was proved in Corollary 3.6. For a general $f \in \mathcal{H}$, notice again that $(f, w - \bar{w}^f)_\mathcal{H} = \mathcal{E}_\alpha(G_\alpha f, w - \bar{w}^f)$, we get

$$\mathcal{E}_\alpha(\bar{w}^f - G_\alpha f, (w - G_\alpha f) - (\bar{w}^f - G_\alpha f)) \geq 0, \quad \forall w \in \mathcal{J}, \quad g \leq w \leq h.$$

Let $\hat{w}^f = \bar{w}^f - G_\alpha f$, $\hat{w} = w - G_\alpha f$, $\hat{g} = g - G_\alpha f$, $\hat{h} = h - G_\alpha f$, we get

$$\mathcal{E}_\alpha(\hat{w}^f, \hat{w} - \hat{w}^f) \geq 0, \quad \forall \hat{w} \in \mathcal{J}, \quad \hat{g} \leq \hat{w} \leq \hat{h},$$

which has a unique solution in view of Corollary 3.6. \square

4 Time Inhomogeneous Stopping Games of Ito Diffusion

In this section we are concerned with a multi-dimensional time inhomogeneous Ito diffusion:

$$dX_t = b(t, X_t)dt + a(t, X_t)dB_t, \quad X_0 = x, \quad (38)$$

where

$$X_t = \begin{pmatrix} X_{1t} \\ \vdots \\ X_{nt} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, a = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}, B_t = \begin{pmatrix} B_{1t} \\ \vdots \\ B_{mt} \end{pmatrix}, m \geq n,$$

and $a_{i,j}, b_i, i = 1, 2, \dots, n, j = 1, 2, \dots, m$, are continuous functions of t and X_t . Define the square matrix $[A_{i,j}] = \mathbf{A} = \frac{1}{2}aa^T$. We assume \mathbf{A} is uniformly non-degenerate, and a, b satisfy the usual Lipschitz conditions so that (38) has a unique strong solution. B_t in (38) is assumed to be the standard multi-dimensional Brownian motion. Thus we are given a system $(\Omega, \mathcal{F}, \mathcal{F}_t, X, \theta_t, P_x)$, where (Ω, \mathcal{F}) is a measurable space, $X = X(\omega)$ is a mapping of Ω into $C(\mathbb{R}^n)$, \mathcal{F}_t is the sigma algebra generated by $X_s(s \leq t)$, and θ_t is a shift operator in Ω such that $X_s(\theta_t\omega) = X_{s+t}(\omega)$. Here $P_x(x \in \mathbb{R})$ is a family of measures under which $\{X_t, t \geq 0\}$ is a diffusion with initial state x .

At each time t , define the infinitesimal generator $L^{(t)}$ as

$$L^{(t)}u(x) = \sum_{i=1}^n b_i(t, x) \frac{\partial u}{\partial x_i} + \sum_{i,j} A_{i,j}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j}. \quad (39)$$

Let the positive Radon measure $\mathbf{m}(dx) = \rho^{(t)}(x)dx$, where $\rho^{(t)}$ satisfies

$$\mathbf{A} \nabla \rho^{(t)} = \rho^{(t)} \mu, \quad \forall t, \quad (40)$$

and $\mu_i = b_i - \sum_{j=1}^n \frac{\partial A_{ji}}{\partial x_j}$, $i = 1, 2, \dots, n$. Notice that when a and b in (38) are constants, $\rho^{(t)}$ reduces to

$$\rho^{(t)}(x) = e^{(\mathbf{A}^{-1}b) \cdot x}.$$

Thus the associated Dirichlet form $(E^{(t)}, F)$ densely embedded in $H = L^2(\mathbb{R}^n; \mathbf{m})$ is then given by

$$E^{(t)}(u, v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \mathbf{A} \nabla v(x) \mathbf{m}(dx), \quad u, v \in F, \quad (41)$$

where

$$F = \{u \in H : u \text{ is continuous, } \|u\|_F^2 = E_1^{(0)}(u, u) < \infty\}.$$

Now we can define the sets $\mathcal{F}, \mathcal{H}, \mathcal{W}$ in the same way as in Section 2, and define the time inhomogeneous Dirichlet form \mathcal{E}_α as well.

Since X_t is a non-degenerate Ito diffusion, the absolute continuity condition on its transition function automatically holds, and for the same reason, the fine and cofine continuity notion can be changed to the usual continuity.

Let $f \in \mathcal{H}, g \in \mathcal{W}$ be continuous functions satisfying the conditions as in Section 3.3, and define the return function $J_z^f(\sigma)$ as in (29), then we have the following result:

Theorem 4.1. *Assume (12) (13) on g and the absolute continuity condition (10) on p_t . Let e_g^f be the solution of*

$$\mathcal{E}_\alpha(e_g^f, \psi - e_g^f) \geq (f, \psi - e_g^f)_{\mathcal{H}}, \quad \forall \psi \in \mathcal{L}_g \cap \mathcal{W}, \quad (42)$$

and let \tilde{e}_g^f be its α -excessive regularization. Then

$$\tilde{e}_g^f(z) = \sup_{\sigma} J_z^f(\sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (43)$$

where $J_z^f(\sigma)$ is defined as (29), and $\tilde{e}_g^f(z)$ is continuous. Furthermore, let the set $B = \{z \in \mathbb{Z} : \tilde{e}_g^f(z) = g(z)\}$ and let σ_B be the first hitting time of B defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = g(Z_{s+t})\}$, then

$$\tilde{e}_g^f(z) = E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})]. \quad (44)$$

For the zero-sum game of Ito diffusion with the return function $J_z^f(\sigma, \tau)$ as defined in (30), we have the following result:

Theorem 4.2. *Let g, h be continuous functions satisfying (22) and (23). Assume (13) on g, h and the absolute continuity condition (10) on p_t . Then there exists a finite and continuous function $\tilde{w}^f \in \mathcal{J}$, $g(z) \leq \tilde{w}^f(z) \leq h(z)$, such that*

$$\mathcal{E}_\alpha(\tilde{w}^f, w - \tilde{w}^f) \geq (f, w - \tilde{w}^f)_{\mathcal{H}}, \quad \forall w \in \mathcal{J}, \quad g \leq w \leq h, \quad (45)$$

and

$$\tilde{w}^f(z) = \sup_{\sigma} \inf_{\tau} J_z^f(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_z^f(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (46)$$

where σ, τ range over all stopping times and $J_z^f(\sigma, \tau)$ is defined in (30). Moreover, the pair $\hat{\tau}, \hat{\sigma}$ defined by

$$\hat{\tau} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = g(Z_{s+t})\}$$

is the saddle point of the game in the sense that

$$J_z^f(\hat{\tau}, \sigma) \leq J_z^f(\hat{\tau}, \hat{\sigma}) \leq J_z^f(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},$$

for all stopping times τ, σ .

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